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RADIAL FRACTIONAL LAPLACE OPERATORS AND HESSIAN INEQUALITIES

FAUSTO FERRARI AND IGOR E. VERBITSKY

Abstract. In this paper we deduce a formula for the fractional Laplace operator $(-\Delta)^s$ on radially symmetric functions useful for some applications. We give a criterion of subharmonicity associated with $(-\Delta)^s$, and apply it to a problem related to the Hessian inequality of Sobolev type:

$$\int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{k}{k+1}} u\right|^{k+1} dx \le C \int_{\mathbb{R}^n} -u\, F_k[u]\, dx,$$

where F_k is the k-Hessian operator on $\mathbb{R}^n$, $1 \le k < \frac{n}{2}$, under some restrictions on a k-convex function u. In particular, we show that the class of u for which the above inequality was established in [FFV] contains the extremal functions for the Hessian Sobolev inequality of X.-J. Wang [W1]. This is proved using logarithmic convexity of the Gaussian ratio of hypergeometric functions which might be of independent interest.

1. Introduction

Let $n \ge 2$. For every $s \in (0,1)$ and a locally integrable function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} \frac{\mid u(x) \mid}{(1+\mid x \mid^2)^{\frac{n+2s}{2}}} dx < +\infty, \tag{1.1}$$

we define the s-fractional Laplace operator as follows:

$$(-\Delta)^s u(x) = c_{s,n} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}} dy, \tag{1.2}$$

where $c_{s,n}$ is a positive normalization constant. The integral is convergent (in the principal value sense if $\frac{1}{2} \le s < 1$) if u is, for instance, a bounded C^2 function. For any u satisfying (1.1), $(-\Delta)^s u$ is defined in the sense of distributions:

$$\langle (-\Delta)^s u,\, h\rangle = \int_{\mathbb{R}^n} u\,(-\Delta)^s\, h\, dx, \tag{1.3}$$

for all test functions $h \in C_0^\infty(\mathbb{R}^n)$. This is a linear nonlocal operator, that is, roughly speaking, if u satisfies $(-\Delta)^s u(x) = 0$ in a domain $\Omega \subseteq \mathbb{R}^n$, then the value of u at any point of Ω depends not only on the neighborhood of the point itself but also on the behavior of the function in the entire space. We refer to [Lan] and [CaS] for further details.

Some new relationships with a class of local nonlinear operators, the k-th Hessian operators, have been pointed out in [FFV]. We recall that the k-th Hessian operator F_k can be defined in several equivalent ways (see e.g. [W2]). For a C^2 function u in a domain $\Omega \subset \mathbb{R}^n$, we define $F_k[u]$, $k \in \mathbb{N}$, $1 \le k \le n$, as the k-th symmetric elementary function of the eigenvalues of $D^2u(x)$, the Hessian matrix of u. Just to give an idea of this family of operators we note that $F_1[u] = \Delta u$ when $k = 1$, and $F_n[u] = \det(D^2u)$ when $k = n$. A function $u \in C^2(\Omega)$ is called k-convex if $F_j[u] \ge 0$ for all $j = 1, 2, \ldots, k$. This definition was extended to general upper semicontinuous

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functions by Trudinger and Wang [TW1], [TW2]. In particular, u is 1-convex if and only if u is subharmonic, while u is n-convex if it is convex in the usual sense.

Clearly, F_k are local operators, since it is possible to calculate $F_k[u]$ pointwise by using second order partial derivatives of u. Moreover, k-th Hessian operators are fully nonlinear while s-fractional Laplace operators are linear. Nevertheless, these operators are closely related. It was shown in [FFV] (Theorem 2.1) that, under certain assumptions on u discussed below, the following inequality holds (see Proposition 5.8 in this paper for the detailed statement):

$$\int_{\mathbb{R}^n} |(-\Delta)^s u|^{k+1}\, dx \le C_{k,n} \int_{\mathbb{R}^n} -u\, F_k[u]\, dx, \tag{1.4}$$

where $s = \frac{k}{k+1}$, $1 \le k < \frac{n}{2}$, and $C_{k,n}$ is positive constant depending only on k and n. The converse inequality holds as well with a different constant, for all k-convex C^2 functions u vanishing at infinity ([FFV], Theorem 3.1).

In this paper we want to improve our knowledge of the s-fractional Laplace operator by studying s-subharmonic functions that are analogous to subharmonic functions for the Laplace operator, and in particular the class of k-convex functions introduced in [FFV] for which (1.4) holds. In fact we will prove below that this class contains extremal functions of the Hessian Sobolev inequality due to X.-J. Wang [W1], [W2]:

$$\left(\int_{\mathbb{R}^n} |u|^q\, dx\right)^{\frac{1}{q}} \le C'_{k,n} \left(\int_{\mathbb{R}^n} -u\, F_k[u]\, dx\right)^{\frac{1}{k+1}}, \tag{1.5}$$

where $1 \le k < \frac{n}{2}$, $q = \frac{n(k+1)}{n-2k}$, and u is a k-convex C^2 function on $\mathbb{R}^n$ vanishing at ∞. We remark that (1.4) implies (1.5) by the classical Sobolev embedding theorem.

Our approach is to study the s-fractional Laplace operator and the corresponding notion of s-subharmonicity for radially symmetric functions $u(x) = u(r)$, $r = |x|$. We will prove the following result.

Theorem 1.1. *Let $s \in (0,1)$. For every radial C^2 function u such that*

$$\int_0^{+\infty} \frac{|\, u(r)\,|}{(1+r)^{n+2s}}\, r^{n-1}\, dr < +\infty, \tag{1.6}$$

the following formula holds:

(1.7)

$$(-\Delta)^s u(r) = c_{s,n}\, r^{-2s} \int_1^{+\infty} \left(u(r) - u(r\tau) + (u(r) - u(\frac{r}{\tau}))\tau^{-n+2s}\right) \tau(\tau^2-1)^{-1-2s} H(\tau) d\tau,$$

where $r = |x| > 0$, $x \in \mathbb{R}^n$, and

$$H(\tau) = 2\pi\alpha_n \int_0^{\pi} \sin^{n-2}\theta\, \frac{(\sqrt{\tau^2 - \sin^2\theta} + \cos\theta)^{1+2s}}{\sqrt{\tau^2 - \sin^2\theta}}\, d\theta, \quad \tau \ge 1, \quad \alpha_n = \frac{\pi^{\frac{n-3}{2}}}{\Gamma(\frac{n-1}{2})}.$$

Clearly, $H(\tau)$ is a positive continuous function on $[1, +\infty)$, with $H(\tau) \simeq \tau^{2s}$ as $\tau \to +\infty$. One can express $H(\tau)$ in terms of the Gaussian hypergeometric function:

$$\tau(\tau^2-1)^{-1-2s} H(\tau) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\, \tau^{-1-2s}\, {}_2F_1(a,b,c,\tau^{-2}) \simeq (\tau-1)^{-1-2s}, \quad \tau \ge 1,$$

where

$$a = \frac{n+2s}{2}, \quad b = 1+s, \quad c = \frac{n}{2}.$$

Note that ${}_2F_1(a,b,c,\tau^{-2})$ has a singularity of order $(\tau-1)^{-1-2s}$ at $\tau = 1$, while $H(\tau)$ is continuous at $\tau = 1$. The integral in (1.7) is convergent for any radial C^2 function u satisfying (1.6).

For every $s \in (0,1]$ and u satisfying (1.1), we say that u is s-subharmonic in Ω if $u : \Omega \to [-\infty, +\infty)$ is upper semicontinuous in Ω, and

$$-(-\Delta)^s u(x) \geq 0 \quad \text{in } D'(\Omega);$$

in other words, $-(-\Delta)^s u$ is a positive Borel measure in Ω.

Theorem 1.2. *Let $s \in (0,1]$. Let $u(r)$ be a radial upper semicontinuous function in $\mathbb{R}_+$ such that (1.6) holds. If, for all $r > 0$ and $\tau \geq 1$,*

$$u(r) - u(r\tau) + (u(r) - u(\frac{r}{\tau}))\tau^{-n+2s} \leq 0, \tag{1.8}$$

then u is s-subharmonic in $\mathbb{R}^n \setminus \{0\}$.

If in Theorem 1.2 one assumes additionally that u is bounded above in a neighborhood of 0, or more generally $\limsup_{r\to 0} u(r)\, r^{n-2s} \leq 0$, then u has a removable singularity at 0 (see [BH], p. 379, Corollary 10.2), i.e., u is s-subharmonic in the entire space $\mathbb{R}^n$ (with $u(0) = \limsup_{r\to 0} u(r)$).

We observe that condition (1.8) is of interest for any real s. The following theorem provides a more convenient pointwise characterization of (1.8). Combined with Theorem 1.2, it gives a useful sufficient condition for a radially symmetric function to be s-subharmonic when $0 < s < 1$.

Theorem 1.3. *Let $s \in \mathbb{R}$. Let $u \in C^2(\mathbb{R}_+)$ be a radial function. Then (1.8) holds if and only if, for all $r > 0$,*

$$u''(r) + (n - 2s + 1)\frac{u'(r)}{r} \geq 0. \tag{1.9}$$

Condition (1.9) is not necessary for a radially symmetric function u to be s-subharmonic when $0 < s < 1$ (see examples below). However, the class of functions obeying (1.9) is quite rich, and contains many interesting s-subharmonic functions.

In the special case $s = \frac{n(k-1)}{2k} + 1$, condition (1.9) coincides with the following characterization of k-convex radially symmetric C^2 functions (see [W1], [W2]):

$$u''(r) + \frac{n-k}{k}\frac{u'(r)}{r} \geq 0, \tag{1.10}$$

where $k = 1, 2, \ldots, n$.

For $\beta > 0$, let us consider

$$f_\beta(x) = -(1 + |\, x \,|^2)^{-\frac{\beta}{2}}, \quad x \in \mathbb{R}^n.$$

It is easy to see (cf. Corollary 4.1 below) that for any $s \in \mathbb{R}$, f_β satisfies (1.8), and consequently (1.9), if and only if $\beta \leq n - 2s$. Moreover, if $0 < s \leq 1$ and $0 < \beta < n$, then f_β is s-subharmonic in $\mathbb{R}^n$ if and only if $\beta \leq n - 2s$.

Now let $\beta = \frac{n}{k} - 2 > 0$, where $k = 1, 2, \ldots, \left[\frac{n}{2}\right]$. Then by (1.10), f_β is k-convex, and is known to be an extremal function for the Hessian Sobolev inequality (1.5); see [W1], [W2]. (A new proof of this inequality, but without the sharp constant, is available; see [V]). Our main goal is to show that $u = f_\beta$ satisfies the following condition introduced in [FFV]:

$$(-\Delta)^s[-(-\Delta)^s u]^k \geq 0 \quad \text{in } \mathbb{R}^n, \tag{1.11}$$

where $s = \frac{k}{k+1}$, $1 \leq k < \frac{n}{2}$. Under this condition the enhanced Hessian Sobolev inequality (1.4) was proved in [FFV] (Theorem 2.1). The following theorem demonstrates that, in particular, this class of functions is nontrivial.

Theorem 1.4. *Let $s = \frac{k}{k+1}$ and $\beta = \frac{n}{k} - 2$, where $1 \leq k \leq \frac{n}{2}$. Then f_β satisfies condition (1.11).*

The proof of Theorem 1.4 for $k \geq 2$ relies on Theorems 1.2 and 1.3, along with a new logarithmic convexity property of the Gaussian quotient of hypergeometric functions $\frac{{}_2F_1(a,b,c,x)}{{}_2F_1(a,b+1,c+1,x)}$.

It is worth observing that when $k = 1$, the function $-u = (-\Delta)^{\frac{1}{2}} f_\beta$ with $\beta = n-2$ ($n \geq 3$), fails to satisfy condition (1.9) with $s = \frac{1}{2}$; however, $-u$ is still $\frac{1}{2}$-subharmonic. The latter is easily checked using the superposition property of fractional Laplacians:

$$(-\Delta)^{\frac{1}{2}}[-(-\Delta)^{\frac{1}{2}} f_\beta] = \Delta f_\beta \geq 0.$$

The paper is organized as follows. The proofs of Theorems 1.1 and 1.2 are given in Section 2. In Section 3 we prove Theorem 1.3 and discuss the s-subharmonicity property for radially symmetric functions. Some examples are given in Section 4. In Section 5 we discuss some further applications of our results. In particular, we prove a Liouville theorem, the maximum principle for radial s-subharmonic functions, and a derivative formula involving the fractional Laplacian. In Section 6, we prove Theorem 1.4 using a convexity property of a certain ratio of hypergeometric functions. Finally, in Section 7, we prove logarithmic convexity of $\frac{{}_2F_1(a,b,c,x)}{{}_2F_1(a,b+1,c+1,x)}$ and convexity of $\frac{{}_2F_1(a,b,c,x)}{{}_2F_1(a+1,b+1,c+1,x)}$ in $(-\infty, 1)$ (Theorem 7.1 and Corollary 7.4) under certain restrictions on the parameters a, b, c using a method developed recently in [KS1] (see also [KS2]). The proofs generalize to ratios of generalized hypergeometric functions ${}_{q+1}F_q$ for $q \geq 2$.

2. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. For the sake of convenience, we will drop the normalization constant $c_{s,n}$ in (1.2) when it does not lead to a confusion. Let u be a radial C^2 function in $\mathbb{R}^n$ satisfying (1.6). If $u(x) = u(|x|)$, $x = rx'$, $y = \rho y'$, and $|\, x' \,|=|\, y' \,| = 1$, then

$$\begin{aligned}
(-\Delta)^s u(x) &= \int_0^{+\infty} \left(\int_{|y'|=1} \frac{u(r) - u(\rho)}{|\, rx' - \rho y' \,|^{n+2s}} \rho^{n-1} dH^{n-1}(y') \right) d\rho \\
&= \int_0^{+\infty} (u(r) - u(\rho))\rho^{n-1} \left(\int_{|y'|=1} \frac{1}{|\, rx' - \rho y' \,|^{n+2s}} dH^{n-1}(y') \right) d\rho \\
&= \int_0^{+\infty} \frac{u(r) - u(\rho)}{r^{n+2s}} \rho^{n-1} \left(\int_{|y'|=1} \frac{1}{|\, x' - \frac{\rho}{r} y' \,|^{n+2s}} dH^{n-1}(y') \right) d\rho \\
&= r^{-1-2s} \int_0^{+\infty} (u(r) - u(r\tau))\tau^{n-1} r \left(\int_{|y'|=1} \frac{1}{|\, x' - \tau y' \,|^{n+2s}} dH^{n-1}(y') \right) d\tau \\
&= \int_0^{+\infty} \frac{u(r) - u(r\tau)}{r^{2s}} \tau^{n-1} \left(\int_{|y'|=1} \frac{1}{|\, x' - \tau y' \,|^{n+2s}} dH^{n-1}(y') \right) d\tau.
\end{aligned}$$

Notice that

$$\int_{|y'|=1} \frac{1}{|\, x' - \tau y' \,|^{n+2s}} dH^{n-1}(y')$$

is independent of $x' \in \{|\, y \,| = 1\}$. Indeed, suppose that $z' \in \{|\, y \,| = 1\}$. Then there exists a unitary matrix Q such that $z' = Qx'$. Thus, performing a change of variables such that $y' = Qw'$, we get:

$$\begin{aligned}
\int_{|y'|=1} \frac{1}{|\, z' - \tau y' \,|^{n+2s}} dH^{n-1}(y') &= \int_{|w'|=1} \frac{1}{|\, Qx' - \tau Qw' \,|^{n+2s}} dH^{n-1}(w') \\
&= \int_{|w'|=1} \frac{1}{|\, x' - \tau w' \,|^{n+2s}} dH^{n-1}(w'),
\end{aligned}$$

because $|\, \det Q \,| = 1$ and $|\, Qv_1 - Qv_2 \,|=|\, v_1 - v_2 \,|$ for every $v_1, v_2 \in \mathbb{R}^n$.

Moreover,

$$\int_{|y'|=1} \frac{1}{\langle z'-\tau y', z'-\tau y'\rangle^{\frac{n+2s}{2}}} dH^{n-1}(y') = \int_{|y'|=1} \frac{1}{(1-2\tau\langle y', z'\rangle+\tau^2)^{\frac{n+2s}{2}}} dH^{n-1}(y')$$
$$= 2\pi\alpha_n \int_0^\pi \frac{\sin^{n-2}\theta}{(1-2\tau\cos\theta+\tau^2)^{\frac{n+2s}{2}}} d\theta,$$

where

$$\alpha_n = \prod_{k=1}^{n-3} \int_0^\pi \sin^k \theta d\theta = \frac{\pi^{\frac{n-3}{2}}}{\Gamma(\frac{n-1}{2})}.$$

Notice also that

$$1-2\tau\cos\theta+\tau^2 \geq 1-2\tau+\tau^2 = (1-\tau)^2.$$

Thus, for every $\tau \neq 1$,

$$\int_0^\pi \frac{\sin^{n-2}\theta}{(1-2\tau\cos\theta+\tau^2)^{\frac{n+2s}{2}}} d\theta$$

is bounded, while for $\tau = 1$,

$$\frac{\sin^{n-2}\theta}{(2-2\cos\theta)^{\frac{n+2s}{2}}} \sim \theta^{n-2}\theta^{-n-2s} \sim \theta^{-2-2s}, \quad \text{as } \theta \to 0.$$

We denote

$$K(\tau) = 2\pi\alpha_n \int_0^\pi \frac{\sin^{n-2}\theta}{(1-2\tau\cos\theta+\tau^2)^{\frac{n+2s}{2}}} d\theta, \tag{2.1}$$

and study the behavior of the integral with respect to τ.

Let $\tau \geq 1$. We perform a change of variable as follows:

$$\frac{\sin\theta}{\sqrt{\tau^2-2\tau\cos\theta+1}} = \frac{\sin\psi}{\tau}.$$

Consequently,

$$\cos\theta = \frac{\sin^2\psi \pm \cos\psi\sqrt{\tau^2-\sin^2\psi}}{\tau}.$$

Hence

$$\begin{aligned} -\sin\theta\frac{d\theta}{d\psi} &= \tau^{-1}\sin\psi\left(2\cos\psi \mp \sqrt{\tau^2-\sin^2\psi} \mp \frac{\cos^2\psi}{\sqrt{\tau^2-\sin^2\psi}}\right) \\ &= \frac{\sin\psi}{\tau\sqrt{\tau^2-\sin^2\psi}}\left(\cos\psi \mp \sqrt{\tau^2-\sin^2\psi}\right)^2 \end{aligned} \tag{2.2}$$

Moreover

$$\begin{aligned} \tau\sin\theta &= \sin\psi\sqrt{\tau^2-2\tau\cos\theta+1} \\ &= \sin\psi\sqrt{\tau^2-2(\sin^2\psi \pm \cos\psi\sqrt{\tau^2-\sin^2\psi})+1} \\ &= \sin\psi\sqrt{\tau^2-\sin^2\psi \mp 2\cos\psi\sqrt{\tau^2-\sin^2\psi}+1-\sin^2\psi} \\ &= \sin\psi\sqrt{\tau^2-\sin^2\psi \mp 2\cos\psi\sqrt{\tau^2-\sin^2\psi}+\cos^2\psi} \\ &= \sin\psi\sqrt{(\sqrt{\tau^2-\sin^2\psi} \mp \cos\psi)^2} = \sin\psi \mid \sqrt{\tau^2-\sin^2\psi} \mp \cos\psi \mid \end{aligned} \tag{2.3}$$

We obtain, by plugging the relation (2.3) into (2.2):

$$\begin{aligned}\frac{d\theta}{d\psi} &= -\frac{1}{\mid \sqrt{\tau^2-\sin^2\psi}\mp\cos\psi \mid \sqrt{\tau^2-\sin^2\psi}}\left(\cos\psi\mp\sqrt{\tau^2-\sin^2\psi}\right)^2 \\ &= -\mid 1\mp\frac{\cos\psi}{\sqrt{\tau^2-\sin^2\psi}}\mid\end{aligned} \tag{2.4}$$

Hence,

$$\begin{aligned}&\int_0^\pi \frac{\sin^{n-2}\theta}{(1-2\tau\cos\theta+\tau^2)^{\frac{n+2s}{2}}}d\theta \\ &= \int_0^\pi (\frac{\sin\psi}{\tau})^{n+2s}\sin^{-2-2s}\theta\left(1\mp\frac{\cos\psi}{\sqrt{\tau^2-\sin^2\psi}}\right)d\psi \\ &= \tau^{2+2s}\int_0^\pi (\frac{\sin\psi}{\tau})^{n+2s}\left(\sin\psi\mid\sqrt{\tau^2-\sin^2\psi}\mp\cos\psi\mid\right)^{-2-2s}\left(1\mp\frac{\cos\psi}{\sqrt{\tau^2-\sin^2\psi}}\right)d\psi \\ &= \tau^{2-n}\int_0^\pi \sin^{n-2}\psi\left(\mid\sqrt{\tau^2-\sin^2\psi}\mp\cos\psi\mid\right)^{-2-2s}\left(1\mp\frac{\cos\psi}{\sqrt{\tau^2-\sin^2\psi}}\right)d\psi \\ &= \tau^{2-n}\int_0^\pi \sin^{n-2}\psi\frac{1}{(\sqrt{\tau^2-\sin^2\psi}\mp\cos\psi)^{1+2s}\sqrt{\tau^2-\sin^2\psi}}d\psi \\ &= \tau^{2-n}(\tau^2-1)^{-1-2s}\int_0^\pi \sin^{n-2}\psi\frac{(\sqrt{\tau^2-\sin^2\psi}+\cos\psi)^{1+2s}}{\sqrt{\tau^2-\sin^2\psi}}d\psi.\end{aligned} \tag{2.5}$$

Let us denote

$$H(\tau) = 2\pi\alpha_n\int_0^\pi \sin^{n-2}\psi\frac{(\sqrt{\tau^2-\sin^2\psi}+\cos\psi)^{1+2s}}{\sqrt{\tau^2-\sin^2\psi}}d\psi. \tag{2.6}$$

Then $H(\tau)=\tau^{n-2}(\tau^2-1)^{1+2s}K(\tau)$. Clearly, $H(\tau)$ is a positive continuous function on $[1,+\infty)$ such that $H(\tau)\simeq\tau^{2s}$ as $\tau\to+\infty$. Moreover, it follows from (2.1) that, as was mentioned in the Introduction, $H(\tau)$ can be expressed in terms of the hypergeometric function ${}_2F_1(a,b,c,\tau)$ (see [MOS], p. 55):

$$H(\tau)=\tau^{-2-2s}(\tau^2-1)^{1+2s}\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\,{}_2F_1(a,b,c,\tau^{-2}),\quad \tau\geq 1,$$

where

$$a=\frac{n+2s}{2},\quad b=1+s,\quad c=\frac{n}{2}.$$

On the other hand

$$\begin{aligned}(-\Delta)^s u(r) &= \int_0^{+\infty}\frac{u(r)-u(r\tau)}{r^{2s}}\tau^{n-1}K(\tau)d\tau \\ &= \int_0^1 \frac{u(r)-u(r\tau)}{r^{2s}}\tau^{n-1}K(\tau)d\tau+\int_1^{+\infty}\frac{u(r)-u(r\tau)}{r^{2s}}\tau(\tau^2-1)^{-1-2s}H(\tau)d\tau.\end{aligned} \tag{2.7}$$

Let us consider the following integral

$$\int_0^1 \frac{u(r)-u(r\tau)}{r^{2s}}\tau^{n-1}K(\tau)d\tau,$$

and perform the following change of variable: $\xi = \frac{1}{\tau}$. Then

$$\begin{aligned}\int_0^1 \frac{u(r)-u(r\tau)}{r^{2s}}\tau^{n-1}K(\tau)d\tau &= \int_1^{+\infty} \frac{u(r)-u(\frac{r}{\xi})}{r^{2s}}\xi^{-n+1}\xi^{-2}K(\frac{1}{\xi})d\xi \\ &= \int_1^{+\infty} \frac{u(r)-u(\frac{r}{\xi})}{r^{2s}}\xi^{-n-1}K(\frac{1}{\xi})d\xi.\end{aligned} \tag{2.8}$$

Notice that

$$K(\frac{1}{\xi}) = 2\pi\alpha_n \xi^{n+2s}\int_0^\pi \frac{\sin^{n-2}\theta}{(\xi^2-2\xi\cos\theta+1)^{\frac{n+2s}{2}}}d\theta.$$

Hence,

(2.9)

$$\begin{aligned}&\int_0^1 \frac{u(r)-u(r\tau)}{r^{2s}}\tau^{n-1}K(\tau)d\tau = \int_1^{+\infty} \frac{u(r)-u(\frac{r}{\xi})}{r^{2s}}\xi^{-n+1}\xi^{-2}K(\frac{1}{\xi})d\xi \\ &= \int_1^{+\infty} \frac{u(r)-u(\frac{r}{\xi})}{r^{2s}}\xi^{-n-1}\xi^{n+2s}K(\xi)d\xi = \int_1^{+\infty} \frac{u(r)-u(\frac{r}{\xi})}{r^{2s}}\xi^{-1+2s}\xi^{2-n}(\xi^2-1)^{-1-2s}H(\xi)d\xi \\ &= \int_1^{+\infty} \frac{u(r)-u(\frac{r}{\xi})}{r^{2s}}\xi^{1-n+2s}(\xi^2-1)^{-1-2s}H(\xi)d\xi.\end{aligned}$$

Thus

$$(-\Delta)^s u(r) = r^{-2s}\int_1^{+\infty}\left(u(r)-u(r\xi)+(u(r)-u(\frac{r}{\xi}))\xi^{-n+2s}\right)\xi(\xi^2-1)^{-1-2s}H(\xi)d\xi. \tag{2.10}$$

The convergence of the integral in (2.10) is discussed below in Sec. 3. ☐

We now deduce Theorem 1.2 from Theorem 1.1 using mollification defined by means of Mellin's convolution which preserves condition (1.8).

Proof of Theorem 1.2. Suppose u is a radially symmetric upper semicontinuous function satisfying the conditions

$$\int_0^{+\infty} \frac{|\, u(r)\,|}{(1+r)^{n+2s}}\, r^{n-1}\, dr < +\infty, \tag{2.11}$$

and

$$u(r)-u(r\xi)+(u(r)-u(\frac{r}{\xi}))\xi^{-n+2s} \le 0, \quad \text{for all } r>0,\ \xi\ge 1. \tag{2.12}$$

Let us show that u is s-subharmonic, i.e., $-(-\Delta)^s u \ge 0$ in the sense of distributions. Let $\phi \in C_0^\infty(\mathbb{R})$ so that $\phi \ge 0$, ϕ is supported in the interval $|r| \le r_0$, where $0 < r_0 < +\infty$, and $\int_{\mathbb{R}} \phi(y)dy = 1$. For $\epsilon > 0$, define the approximate identity on $\mathbb{R}_+$ by:

$$\phi_\epsilon(r) = \frac{1}{\epsilon}\phi\left(\frac{\log r}{\epsilon}\right), \quad r>0. \tag{2.13}$$

Then clearly, for every $\epsilon > 0$,

$$\int_0^{+\infty} \phi_\epsilon(r)\frac{dr}{r} = 1. \tag{2.14}$$

We observe that (2.12) is invariant under the Mellin convolution:

$$u_\epsilon(\tau) = \int_0^{+\infty} \phi_\epsilon\left(\frac{\tau}{t}\right)u(t)\frac{dt}{t}, \quad \tau>0. \tag{2.15}$$

Indeed, integrating both sides of (2.12) against $\phi_\epsilon(\frac{\tau}{r})\frac{dr}{r}$, we obtain:

$$u_\epsilon(\tau) - u_\epsilon(\tau\xi) + (u_\epsilon(\tau) - u_\epsilon(\frac{\tau}{\xi}))\xi^{-n+2s} \le 0, \quad \text{for all } \tau > 0,\ \xi \ge 1. \tag{2.16}$$

Moreover, obviously $u_\epsilon \in C^\infty(\mathbb{R}_+)$, and

$$\int_0^{+\infty} \frac{|u_\epsilon(r)|\, r^{n-1}}{(1+r)^{n+2s}} dr \le C \int_0^{+\infty} \frac{|u(r)|\, r^{n-1}}{(1+r)^{n+2s}} dr < +\infty, \tag{2.17}$$

where $C = C(\epsilon, r_0, s, n)$ is a positive constant. Indeed, by Fubini's theorem,

$$\int_0^{+\infty} \frac{|u_\epsilon(r)|\, r^{n-1}}{(1+r)^{n+2s}} dr \le \int_0^{+\infty} |u(t)| \int_0^{+\infty} \frac{r^n}{(1+r)^{n+2s}} \phi_\epsilon\big(\frac{r}{t}\big) \frac{dr}{r}\, \frac{dt}{t}$$

$$= \int_0^{+\infty} |u(t)| \int_0^{+\infty} \frac{\lambda^n t^n}{(1+\lambda t)^{n+2s}} \phi_\epsilon\big(\lambda\big) \frac{d\lambda}{\lambda}\, \frac{dt}{t} \le C \int_0^{+\infty} \frac{|u(t)|\, t^n}{(1+t)^{n+2s}} \frac{dt}{t} < +\infty,$$

where the last estimate follows since $\phi_\epsilon(\lambda)$ vanishes outside the interval $(e^{-\epsilon r_0},\, e^{\epsilon r_0})$, and (2.14) holds.

Let $h \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ be a nonnegative test function supported in $0 < |x| \le R < +\infty$. We observe that, since h is compactly supported, it follows from (1.2) that $(-\Delta)^s h \in C^\infty(\mathbb{R}^n)$, and

$$|(-\Delta)^s h(x)| \le \frac{C}{(1+|x|)^{n+2s}}, \quad x \in \mathbb{R}^n. \tag{2.18}$$

In particular,

$$\langle (-\Delta)^s u,\, h\rangle = \int_{\mathbb{R}^n} u\, (-\Delta)^s\, h\, dx$$

is well-defined in terms of distributions (see [Lan], Sec. 1.6). By Theorem 1.1 and (2.16), $(-\Delta)^s u_\epsilon \ge 0$, and hence, using (2.11) and recalling that $u_\epsilon \in C^\infty(\mathbb{R}^n \setminus \{0\})$, we obtain:

$$\langle (-\Delta)^s u_\epsilon,\, h\rangle = \langle u_\epsilon,\, (-\Delta)^s h\rangle = \int_{\mathbb{R}^n} u_\epsilon\, (-\Delta)^s h\, dx \ge 0, \tag{2.19}$$

for every $h \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $h \ge 0$.

It remains to prove the approximation property:

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} u_\epsilon\, (-\Delta)^s\, h\, dx = \int_{\mathbb{R}^n} u\, (-\Delta)^s h\, dx. \tag{2.20}$$

Denote by $\psi(t)$ the spherical mean of $(-\Delta)^s h$:

$$\psi(t) = \int_{|x'|=1} (-\Delta)^s h(tx')\, dH^{n-1}(x'), \quad t > 0.$$

Then

$$\int_{\mathbb{R}^n} [u(x) - u_\epsilon(x)]\, (-\Delta)^s h(x)\, dx = \int_0^{+\infty} [u(t) - u_\epsilon(t)]\, \psi(t)\, t^n \frac{dt}{t}.$$

We estimate:

$$\left|\int_0^{+\infty} [u(t) - u_\epsilon(t)]\psi(t) t^n \frac{dt}{t}\right| = \left|\int_0^{+\infty} u(t) \left[\psi(t)t^n - \int_0^{+\infty} \phi_\epsilon\big(\frac{r}{t}\big)\psi(r) r^n \frac{dr}{r}\right] \frac{dt}{t}\right|$$

$$\le \int_0^{+\infty} |u(t)| \int_0^{+\infty} |t^n\psi(t) - r^n\psi(r)|\, \phi_\epsilon\big(\frac{r}{t}\big) \frac{dr}{r}\, \frac{dt}{t}.$$

We recall that, since $\phi(y)$ is supported in the interval $|y| \le r_0$, it follows that $\phi_\epsilon\big(\frac{r}{t}\big)$ is supported in the interval where $|\log \frac{r}{t}| \le \epsilon r_0$. Hence, in the above estimates we can assume:

$$e^{-\epsilon r_0} - 1 \le \frac{r}{t} - 1 \le e^{\epsilon r_0} - 1.$$

From this we deduce:

$$|r-t| \le \delta t, \tag{2.21}$$

where $\delta = e^{\epsilon r_0} - 1 \to 0$ as $\epsilon \to 0$.

We notice that, since $(-\Delta)^s h \in C^\infty(\mathbb{R}^n)$, it follows that its spherical mean ψ is infinitely differentiable on $[0, +\infty)$ (see, e.g., [Lan], Sec. I.6). We will need the following estimates:

$$|\psi(t)| \le \frac{C}{(1+t)^{n+2s}}, \quad |\psi'(t)| \le \frac{C}{(1+t)^{n+2s+1}}, \quad t \ge 0, \tag{2.22}$$

where C depends on h. Indeed, $h(x) = 0$ for $|x| > R$, and consequently (1.2) yields, for $t > R$:

$$\psi(t) = -c_{s,n} \int_{|y|\le R} h(y) \int_{|x'|=1} \frac{1}{|y - tx'|^{n+2s}} dH^{n-1}(x')\, dy,$$

$$\psi'(t) = c_{s,n}(n+2s) \int_{|y|\le R} h(y) \int_{|x'|=1} \frac{t - x'\cdot y}{|y-tx'|^{n+2s+2}} dH^{n-1}(x')\, dy$$

$$= c_{s,n}(n+2s)\, t \int_{|y|\le R} h(y) \int_{|x'|=1} \frac{1}{|y-tx'|^{n+2s+2}} dH^{n-1}(x')\, dy.$$

Here $|y - tx'| \ge t - R > \frac{t}{2}$ for $t > 2R$, which yields (2.22) for $t > 2R$. Since ψ is infinitely differentiable on $[0, +\infty)$, it follows that (2.22) holds for all $t \ge 0$.

Let $\psi_1(t) = t^n \psi(t)$. Clearly, estimates (2.22) yield:

$$|\psi_1(t)| \le \frac{C\, t^n}{(1+t)^{n+2s}}, \quad |\psi_1'(t)| \le \frac{C\, t^{n-1}}{(1+t)^{n+2s}}, \quad t \ge 0. \tag{2.23}$$

Invoking the mean value inequality we estimate:

$$|t^n\psi(t) - r^n\psi(r)| = |\psi_1(t) - \psi_1(r)| \le |r-t|\, |\psi_1'(\lambda)| \le C\, |r-t|\, \frac{\lambda^{n-1}}{(1+\lambda)^{n+2s}},$$

for some λ between t and r. Assuming by (2.21) that $|r-t| < \delta t$, we see that $t(1-\delta) \le \lambda \le t(1+\delta)$. Combining the preceding estimates, we obtain:

$$|t^n\psi(t) - r^n\psi(r)| \le \frac{C\,\delta\,(1+\delta)^{n-1}}{(1-\delta)^{n+2s}} \, \frac{t^n}{(1+t)^{n+2s}}.$$

Using this together with (2.14), we conclude:

$$\int_0^{+\infty} |u(t)| \int_0^{+\infty} |t^n\psi(t) - r^n\psi(r)|\, \phi_\epsilon\big(\frac{r}{t}\big) \frac{dr}{r}\, \frac{dt}{t} \le \frac{C\,\delta\,(1+\delta)^{n-1}}{(1-\delta)^{n+2s}} \int_0^{+\infty} |u(t)|\, \frac{t^{n-1}}{(1+t)^{n+2s}} dt,$$

where the right-hand side is finite by (2.11). Letting $\epsilon \to 0$, and hence $\delta \to 0$, we conclude the proof of the approximation property (2.20). $\square$

3. Fractional subharmonicity for radially symmetric functions

In this section we study further the condition

$$u(r) - u(r\xi) + (u(r) - u(\frac{r}{\xi}))\xi^{-n+2s} \le 0, \quad \text{for all } r > 0,\ \xi \ge 1, \tag{3.1}$$

which ensures that a radially symmetric C^2 function u is s-subharmonic, i.e., $-(-\Delta)^s u \ge 0$, for $0 < s \le 1$. However, this condition makes sense for any real s. We will show that for C^2 functions u it is equivalent to:

$$u''(r) + (n+1-2s)\frac{u'(r)}{r} \ge 0, \quad \text{for all } r > 0, \tag{3.2}$$

which is stated as Theorem 1.3 in the Introduction.

Suppose u is a radial C^2 function. It is worth noting that in a neighborhood of 1, whenever $\xi \to 1$, $\xi \geq 1$,

$$u(r) - u(r\xi) + (u(r) - u(\frac{r}{\xi}))\xi^{-n+2s} = -u'(r)r\frac{(\xi-1)^2(\xi^{n+1-2s}-1)}{\xi^{n+1-2s}(\xi-1)} - r^2\frac{u''(r)}{2}(\xi-1)^2(1+\frac{1}{\xi^{n+2-2s}})$$

$$+\,\xi^{-n+2s}o((\xi-1)^2) = -\frac{1}{2}r^2\frac{(\xi-1)^2(\xi^{n+2-2s}+1)}{\xi^{n+2-2s}}(u''(r) + \frac{2\xi(\xi^{n+1-2s}-1)}{r(1+\xi^{n+2-2s})(\xi-1)}u'(r)) + o(\xi-1)^2$$

$$= -\frac{1}{2}r^2\frac{(\xi-1)^2(\xi^{n+2-2s}+1)}{\xi^{n+2-2s}}\mathcal{L}u,$$

where

$$\mathcal{L}u = u'' + \frac{2\xi(n+1-2s+\frac{n-2s}{2}(\xi-1)+\frac{n-1-2s}{6}(\xi-1)^2+o(\xi-1)^2)}{(1+\xi^{n+2-2s})}\frac{u'}{r} + o(r,(\xi-1)^2)$$

$$= u'' + (\frac{2\xi(n+1-2s)}{1+\xi^{n+2-2s}} + g(\xi-1))\frac{u'}{r}$$

$$= u'' + (n+1-2s)\frac{u'}{r}$$

$$+\,\frac{(n+1-2s)(2\xi-1-\xi^{n+2-2s})+\frac{n-2s}{2}(\xi-1)+\frac{n-1-2s}{6}(\xi-1)^2\frac{u'}{r}+o(\xi-1)^2)}{(1+\xi^{n+2-2s})}$$

$$+\,o(r,(\xi-1)^2) = u'' + (n+1-2s)(1+h(\xi-1))\frac{u'}{r},$$

where $\lim_{\xi\to 1^+}\frac{h(\xi-1)}{\xi-1} = 1$.

In particular, the integral in (1.7) is convergent in a neighborhood of 1, since for any fixed $r > 0$ it behaves as $(1-\xi)^{1-2s}$ that converges whenever $0 < s < 1$. Moreover, by (2.11) it converges when $\xi \to +\infty$. Hence the integral in (1.7) is convergent for every C^2 function u satisfying (2.11).

The above calculation demonstrates that condition (3.1) is closely related to condition (3.2) for any $s \in \mathbb{R}$, which is proved below.

Proof of Theorem 1.3. Let $s \in \mathbb{R}$. Suppose $r > 0$ and $\xi \geq 1$. Dividing both sides of (3.1) by $(\xi-1)^2$ and passing to the limit as $\xi \to 1^+$, we deduce (3.2).

Conversely, suppose that (3.2) holds, and $u'(r) \geq 0$. Let $\gamma = n - 2s$. Making a substitution $r = e^t$, $\xi = e^x$, where $t \in \mathbb{R}$, and $x \geq 0$, we rewrite (3.1) in the equivalent form:

$$(v(t+x)-v(t))e^{\gamma x} + v(t-x) - v(t) \geq 0, \quad \text{for all } t \in \mathbb{R},\ x \geq 0, \tag{3.3}$$

where $v(t) = u(e^t)$, while (3.2) is equivalent to

$$v''(t) + \gamma v'(t) \geq 0, \quad \text{for all } t \in \mathbb{R}. \tag{3.4}$$

We next fix $t \in \mathbb{R}$, and let $\phi(x) = v(x+t) - v(t)$, where $x \in \mathbb{R}$. Then clearly $\phi(0) = 0$, $\phi'(0) = v'(t)$, and (3.4) is equivalent to

$$\phi''(x) + \gamma\phi'(x) \geq 0, \quad \text{for all } x \in \mathbb{R}. \tag{3.5}$$

We need to prove:

$$\phi(x)e^{\gamma x} + \phi(-x) \geq 0, \quad x \geq 0, \tag{3.6}$$

which is equivalent to (3.3). It is not difficult to deduce (3.6) from (3.5) using Gronwall's inequality (see [Har], Sec. 3.1, Theorem 1.1) in the case $\gamma > 0$.

Let us prove this directly for all $\gamma \in \mathbb{R}$. By (3.5), it folows that $\phi'(x)+\gamma\phi(x)$ is non-decreasing on $\mathbb{R}$. Since $\phi(0) = 0$, we obtain:

$$\phi'(x) + \gamma\phi(x) \geq \phi'(0), \quad \text{for all } x \geq 0. \tag{3.7}$$

Let

$$\psi(x) = \phi(-x) + \phi(x)e^{\gamma x}, \quad x \in \mathbb{R}.$$

We next show that $\psi'(x) \geq 0$ for all $x \geq 0$. By (3.7),

$$\psi'(x) = -\phi'(-x) + \big(\phi'(x) + \gamma\phi(x)\big)\, e^{\gamma x} \geq -\phi'(-x) + \phi'(0)e^{\gamma x}, \quad \text{for all } x \geq 0.$$

It remains to prove the inequality

$$\phi'(-x) \leq \phi'(0)\, e^{\gamma x}, \quad \text{for all } x \geq 0. \tag{3.8}$$

Let $g(x) = e^{-\gamma x}\phi'(-x)$. Then

$$g'(x) = -\gamma e^{-\gamma x}\phi'(-x) - e^{-\gamma x}\phi''(-x) = -e^{-\gamma x}(\phi''(-x) + \gamma\phi'(-x)) \leq 0,$$

by (3.5). Hence g is nonincreasing, and consequently $g(x) \leq g(0)$ for all $x \geq 0$. This proves (3.8). Thus $\psi'(x) \geq 0$ for all $x \geq 0$, and since $\psi(0) = 0$, we deduce $\psi(x) \geq 0$ for all $x \geq 0$. This proves (3.6), which in its turn yields (3.1).

□

4. Examples of s−subharmonic functions and further remarks

We consider the fundamental solution of the k-Hessian operator F_k,

$$u(x) = - \mid x \mid^{-(\frac{n}{k}-2)}, \quad x \in \mathbb{R}^n,$$

for $1 \leq k \leq \frac{n}{2}$, which satisfies the equation

$$F_k[u] = \delta_0,$$

in the viscosity sense (see [TW2], [W2]).

It is not difficult to verify directly that u is a radial function satisfying the condition

$$u(r) - u(r\xi) + (u(r) - u(\frac{r}{\xi}))\xi^{-n+2s} \leq 0, \quad \text{for all } r > 0,\ \xi \geq 1, \tag{4.1}$$

if $s = \frac{n(k-1)}{2k} + 1$, or equivalently $n - 2s = \frac{n}{k} - 2$.

More generally, for $\beta > 0$, let us consider the function

$$u(x) = - \mid x \mid^{-\beta}, \quad x \in \mathbb{R}^n.$$

If $\beta = n - 2s$, we have:

$$\begin{aligned} \frac{u(r) - u(r\xi)}{u(\frac{r}{\xi}) - u(r)} &= \frac{-r^{-(n-2s)} + r^{-(n-2s)}\xi^{-(n-2s)}}{-r^{-(n-2s)}\xi^{n-2s} + r^{-(n-2s)}} \\ &= \frac{-1 + \xi^{-(n-2s)}}{-\xi^{n-2s} + 1} = \xi^{-(n-2s)}. \end{aligned}$$

It follows that, for $\beta = n-2s$ and $s < \frac{n}{2}$, condition (4.1) turns into an equality for all $r > 0$, $\xi \geq 1$. Unfortunately the k-energy of $u(x) = - \mid x \mid^{-(\frac{n}{k}-2)}$ is unbounded, and so such functions cannot be used in Hessian inequalities of the type (1.4) or (1.5) discussed above.

This suggests considering the function

$$f_\beta(x) = -(1 + \mid x \mid^2)^{-\frac{\beta}{2}}, \quad x \in \mathbb{R}^n,$$

for $\beta > 0$. When $\beta = \frac{n}{k} - 2$, f is a k-convex function of finite k-energy. Moreover, f_β is known to be an extremal function for the Hessian Sobolev inequality (1.5) (see [W1], [W2]).

Corollary 4.1. *Let $s \in \mathbb{R}$ and $\beta > 0$. Then f_β satisfies (1.8), and consequently (1.9), if and only if $\beta \leq n - 2s$. Moreover, if $0 < s \leq 1$ and $0 < \beta < n$, then f_β is s-subharmonic in $\mathbb{R}^n$ if and only if $\beta \leq n - 2s$.*

Proof. Let $r = |x|$. It is easy to see that

$$f'_\beta(r) = -\beta r(r^2+1)^{-\frac{\beta}{2}-1}, \quad f''_\beta(r) = \beta(r^2+1)^{-\frac{\beta}{2}-2}\left[(\beta+1)r^2-1\right].$$

Hence,

$$f''_\beta(r) + \frac{n-2s+1}{r} f'_\beta(r) = \beta(r^2+1)^{-\frac{\beta}{2}-2}\left[(\beta-n+2s)r^2-(n-2s+2)\right].$$

Suppose $0 < \beta \le n-2s$. Then the right-hand side of the preceding inequality is negative, and (1.9) holds. If $0 < s \le 1$, this implies that f_β is s-subharmonic.

If $\beta > n-2s$, then $f''_\beta(r) + \frac{n-2s+1}{r} f'_\beta(r)$ is either positive or changes sign. Hence, (1.9) fails in this case.

Moreover, by formula (6.5) discussed below,

$$\phi(r) = -(-\Delta)^s f_\beta(r) = C(\alpha,\beta,n)\, F(a,b,c,-r^2),$$

where $C(\alpha,\beta,n)$ is a positive constant, and

$$a = \frac{n+2s}{2}, \quad b = \frac{2s+\beta}{2}, \quad c = \frac{n}{2}, \tag{4.2}$$

where $F(x) ={}_2F_1(a,b,c,x)$ is the hypergeometric function. Notice that $F(0) = 1$ and by a formula due to Gauss (see [AAR], Theorem 2.2.2, p. 66):

$$F(1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \text{if } c > a+b. \tag{4.3}$$

By Pfaff's transformation,

$$F(a,b,c,-r^2) = (1+r^2)^{-b} F(c-a,b,c,\frac{r^2}{r^2+1}).$$

It follows:

$$F(-r^2) \sim (1+r^2)^{-b}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-\beta}{2})}{\Gamma(\frac{n+s}{2})\Gamma(\frac{n-\beta}{2}-s)} \quad \text{as } r \to +\infty. \tag{4.4}$$

Hence, when $n-2s < \beta < n$, $0 < s < 1$, $\phi(r) < 0$ for r large, and so changes sign. In other words, f_β fails to be s-subharmonic in this case. □

5. Applications

We first recall the following definitions. Let $s \in (0,1]$. Let u be any continuous (or more generally upper semicontinuous) function $u : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ such that

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|^2)^{\frac{n+2s}{2}}} dx < +\infty.$$

We say that u is s-subharmonic in $\mathbb{R}^n$ if

$$-(-\Delta)^s u \ge 0 \quad \text{in } D'(\mathbb{R}^n).$$

Analogously we shall say that u is s-superharmonic in $\mathbb{R}^n$

$$-(-\Delta)^s u \le 0 \quad \text{in } D'(\mathbb{R}^n).$$

Whenever u is both s-subharmonic and s-superharmonic in we shall say that u is s-harmonic in $\mathbb{R}^n$.

We remark that from the representation given in Theorem 1.1 we can deduce a Liouville theorem for radial s-subharmonic functions.

Corollary 5.1. *Assume that u is a positive continuous function, radially decreasing and $s \in (0,1]$. Suppose that*

$$\int_{\mathbb{R}^n} \frac{\mid u(x) \mid}{(1+\mid x \mid^2)^{\frac{n+2s}{2}}} dx < +\infty,$$

and $-(-\Delta)^s u \geq 0$. If $\lim_{r\to+\infty} u(r) = 0$ then $u = 0$.

Proof. We remark that if u is radially decreasing then $u(r) \geq u(r\xi)$ and $u(r) \leq u(\frac{r}{\xi})$ for every $\xi \geq 1$. Hence, recalling Theorem 1.2 inequality 1.8, for every $\xi \geq 1$ and for every positive r it results

$$\frac{u(r) - u(\frac{r}{\xi})}{u(r\xi) - u(r)} \geq \xi^{n-2s}.$$

On the other hand, for every fixed r, if $\xi \to +\infty$, then

$$\lim_{\xi\to+\infty} \frac{u(r) - u(\frac{r}{\xi})}{u(r\xi) - u(r)} = +\infty.$$

We recall that $\lim_{r\to+\infty} u(r) = 0$, so we get

$$\lim_{\xi\to+\infty} \frac{u(r) - u(0)}{-u(r)} = +\infty,$$

which implies a contradiction whenever u is bounded and not identically zero. □

The following result gives a derivative rule for fractional Laplacians.

Theorem 5.2. *Let $s \in [0,1]$. For every differentiable radial function u such that*

$$\int_{\mathbb{R}^n} \frac{\mid u(x) \mid}{(1+\mid x \mid^2)^{\frac{n+2s}{2}}} dx < +\infty,$$

the following formula holds:

$$\frac{d(-\Delta)^s u(r)}{dr} = \frac{1}{r}(-\Delta)^s(-2su + ru'). \tag{5.1}$$

Proof. Differentiating (1.7), we obtain:

$$\begin{aligned}
&\frac{d(-\Delta)^s u(r)}{dr} = -\frac{2s}{r}(-\Delta)^s u(r) \\
&+ r^{-1-2s} \int_1^{+\infty} \left((u'(r) - u'(r\xi)\xi + (u'(r) - \frac{u'(\frac{r}{\xi})}{\xi})\xi^{-n+2s} \right) \xi(\xi^2-1)^{-1-2s} H(\xi) d\xi \\
&= -\frac{2s}{r}(-\Delta)^s u(r) \\
&+ r^{-2s} \int_1^{+\infty} \left(r^{-1} \left((ru'(r) - u'(r\xi)r\xi) + (ru'(r) - \frac{u'(\frac{r}{\xi})}{\frac{\xi}{r}}) \right) \xi^{-n+2s} \right) \xi(\xi^2-1)^{-1-2s} H(\xi) d\xi \\
&= -\frac{2s}{r}(-\Delta)^s u(r) + \frac{1}{r}(-\Delta)^s(ru'(r)) = \frac{1}{r}(-\Delta)^s(-2su + ru'),
\end{aligned}$$

□

If

$$f_k(x) = -(1+r^2)^{-(\frac{n}{2k}-1)},$$

then

$$f_k'(r) = 2(\frac{n}{2k} - 1)(1+r^2)^{-\frac{n}{2k}} r$$

and recalling formula (5.1) we get

$$
\begin{aligned}
&-2sf_k(r) + rf_k'(r) = 2s(1+r^2)^{-(\frac{n}{2k}-1)} + 2(\frac{n}{2k}-1)(1+r^2)^{-\frac{n}{2k}}r^2 \\
=&2(1+r^2)^{-\frac{n}{2k}}\left(s(1+r^2)+(\frac{n}{2k}-1)r^2\right) = 2(1+r^2)^{-\frac{n}{2k}}\left(s+(s+\frac{n}{2k}-1)r^2\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{d(-\Delta)^s f_k(r)}{dr} = \frac{1}{r}(-\Delta)^s(-2sf_k + rf_k') = \frac{2}{r}(-\Delta)^s\left((1+r^2)^{-\frac{n}{2k}}\left(s+(s+\frac{n}{2k}-1)r^2\right)\right) \\
&= \frac{2}{r}(-\Delta)^s\left((1+r^2)^{-\frac{n}{2k}}\left((s+\frac{n}{2k}-1)+(s+\frac{n}{2k}-1)r^2\right)\right) \\
&+ \frac{2}{r}(-\Delta)^s\left((1+r^2)^{-\frac{n}{2k}}\left(s-(s+\frac{n}{2k}-1)\right)\right) \\
&= \frac{2(s+\frac{n}{2k}-1)}{r}(-\Delta)^s\left((1+r^2)^{-(\frac{n}{2k}-1)}\right) - \frac{2(\frac{n}{2k}-1))}{r}(-\Delta)^s\left((1+r^2)^{-\frac{n}{2k}}\right).
\end{aligned}
$$

In the next theorem we deduce the maximum principle for radial s-subharmonic functions.

Theorem 5.3. *Suppose that u is a radial s-subharmonic function $s \in (0,1]$. Then either u is constant or u can not realize a maximum in $\mathbb{R}^n$.*

Proof. Indeed, by contradiction, suppose that u realizes an absolute maximum in $\bar r \in \mathbb{R}^n$. Hence for every $\xi \geq 1$, $u(\bar r) - u(\bar r\xi) \geq 0$, $u(\bar r) - u(\frac{\bar r}{\xi}) \geq 0$ and as a consequence

$$u(\bar r) - u(\bar r\xi) + \left(u(\bar r) - u(\frac{\bar r}{\xi})\right)\xi^{2s-n} \geq 0.$$

On the other hand

$$-(-\Delta)^s(\bar r) \geq 0,$$

hence for every $\xi \geq 0$

$$u(\bar r) - u(\bar r\xi) + \left(u(\bar r) - u(\frac{\bar r}{\xi})\right)\xi^{2s-n} = 0$$

and in particular this implies $u \equiv u(\bar r)$ because $u(\bar r) - u(\bar r\xi) \geq 0$ and $(u(\bar r) - u(\frac{\bar r}{\xi})) \geq 0$. $\square$

Corollary 5.4. *If u is a radial s-harmonic function, then either u is constant or u does not realize either an absolute maximum or an absolute minimum.*

Corollary 5.5. *Suppose u_1 and u_2 are two radial s-harmonic functions. If $u_1 \geq u_2$, then $u_1 > u_2$, or $u_1 \equiv u_2$, i.e., u_1 cannot intersect with u_2; otherwise u_1 coincides with u_2.*

Proof. Indeed, if there exists $\bar r$ such that $u_1(\bar r) = u_2(\bar r)$, then $u_1 - u_2$ has a minimum in $\bar r$ and $u_1 - u_2$ is s-harmonic. Then $u_1 \equiv u_2$ by the previous result. $\square$

Corollary 5.6. *Suppose that u is a radial continuous s-harmonic function vanishing at infinity. Then $u \equiv 0$.*

Proof. If $\sup_{\mathbb{R}^n} u = 0$ then u realizes its minimum in $\mathbb{R}^n$. Then we can apply Corollary 5.4 obtaining that u is constant and necessarely $u \equiv 0$. Analogously if $\inf_{\mathbb{R}^n} u = 0$ then u has to attain its maximum in $\mathbb{R}^n$ otherwise $u \equiv 0$. Thus by Corollary 5.4 we conclude that $u \equiv 0$ indeed. $\square$

Corollary 5.7. *Let u_1, u_2 be continuous radially symmetric functions such that*

$$(-\Delta)^s u_1 = (-\Delta)^s u_2.$$

If $\lim_{r\to+\infty}(u_1 - u_2) = 0$ then $u_1 \equiv u_2$.

Proof. Recalling the linearity of the fractional Laplace operator we get:

$$(-\Delta)^s(u_1 - u_2) = 0.$$

Moreover $\lim_{r\to+\infty}(u_1 - u_2) = 0$. It follows by Corollary 5.7 that $u_1 \equiv u_2$. □

Our results make it possible to provide an answer to a question which was left open in [FFV]. More precisely, in [FFV], Theorem 2.1, an inequality involving k-convex functions and the fractional Laplace operator was proved. For reader's convenience we state below the result in [FFV].

Proposition 5.8 (Ferrari-Franchi-Verbitsky)**.** $1 \le k < \frac{n}{2}$*, and let* $\alpha = \frac{2k}{k+1}$. *Suppose* $u \in C^2(\mathbb{R}^n)$ *is a* $k-$*convex function on* $\mathbb{R}^n$ *vanishing at* ∞. *If*

(i) $-(-\Delta)^{\alpha/2}u \ge 0$,
(ii) $(-\Delta)^{\alpha/2}[-(-\Delta)^{\alpha/2}u]^k \ge 0$,

then there exists a positive constant $C_{k,n}$ *such that*

$$\int_{\mathbb{R}^n} \left(-(-\Delta)^{\alpha/2}u\right)^{k+1} dx \le C_{k,n} \int_{\mathbb{R}^n} -u\, F_k[u]\, dx. \tag{5.2}$$

It was unclear if, for $1 \le k < \frac{n}{2}$, the set

$$\mathcal{FC}_k = \{u \in C^2(\mathbb{R}^n) : \ u, \ k-\text{convex}, \ \text{vanishing at } \infty, \text{(i) and (ii) hold}\},$$

was not trivial. Obviously is not empty because for any k, $1 \le k < \frac{n}{2}$, $0 \in \mathcal{FC}_k$.

Theorems 1.2 and 1.3 in this paper will be employed to give a positive answer to this question. Indeed,

$$f_k(x) = -(1+ \mid x \mid^2)^{-(\frac{n}{2k}-1)}$$

is k-convex, vanishing at ∞, and $-(-\Delta)^{\alpha/2}f_k \ge 0$, for every $k \in \mathbb{N}$ such that $\frac{n}{2} \ge k$, i.e. f_k satisfies (i). Moreover if $k = 1$, then $\alpha = 1$, and condition (ii) becomes

$$(-\Delta)^{1/2}[-(-\Delta)^{1/2}f_1] = \Delta f_1 \ge 0.$$

The case $k \ge 2$, which is much harder, is considered in the next two sections.

6. The iterated fractional Laplacian condition

In this section we verify the iterated fractional Laplacian condition of Proposition 5.8:

$$(-\Delta)^{\alpha/2}[-(-\Delta)^{\alpha/2}u]^k \ge 0, \tag{6.1}$$

where $\alpha = \frac{2k}{k+1}$, $1 \le k < \frac{n}{2}$, for the k-convex function $u(x) = -(1+|x|^2)^{-\beta/2}$ on $\mathbb{R}^n$ with $\beta = \frac{n}{k} - 2 > 0$. As explained above, the case $k = 1$ is trivial, and from now on we will assume $k \ge 2$. We observe that u is an extremal function in the important Hessian Sobolev inequality of X.-J. Wang [W1], [W2]. The Fourier transform of u is given by the radially symmetric function

$$\hat{u}(\xi) = -C(\beta, n)\frac{K_{\frac{n-\beta}{2}}(|\xi|)}{|\xi|^{\frac{n-\beta}{2}}}, \quad \xi \in \mathbb{R}^n, \tag{6.2}$$

where $C(\beta, n)$ is a positive constant, and K_γ is the modified Bessel function of order γ.

It is possible to express $(-\Delta)^{\alpha/2}u$ in terms of the Gaussian hypergeometric function

$$F(a, b, c, x) = {}_2F_1(a, b, c, x)$$

discussed in the next section. For radially symmetric functions $f(r)$, where $r = |x|$, the Fourier transform formula can be stated in the form:

$$\hat{f}(s) = (2\pi)^{n/2} \int_0^{+\infty} \frac{J_{\frac{n-2}{2}}(sr)}{(sr)^{\frac{n-2}{2}}} f(r)\, r^{n-1}\, dr, \tag{6.3}$$

where $s=|\xi|$. Hence, from the preceding formulas we deduce:

$$-(-\Delta)^{\alpha/2}u(s)=C(\alpha,\beta,n)\,s^{\frac{2-n}{2}}\int_0^{+\infty}K_{\frac{n-\beta}{2}}(r)\,J_{\frac{n-2}{2}}(sr)\,r^{\frac{\beta}{2}+\alpha}\,dr, \tag{6.4}$$

where $C(\alpha,\beta,n)$ is a positive constant.

Using the well-known integral involving the product of Bessel functions J_γ and modified Bessel functions K_γ, we obtain the explicit formula for $(-\Delta)^{\alpha/2}u$ (see e.g., [MOS], Sec. 3.8, p. 100):

$$-(-\Delta)^{\alpha/2}u(x)=C(\alpha,\beta,n)\,F(a,b,c,-|x|^2),\quad x\in\mathbb{R}^n, \tag{6.5}$$

where $C(\alpha,\beta,n)$ is a positive constant. The parameters a, b, and c are given by:

$$a=\frac{n+\alpha}{2},\quad b=\frac{\alpha+\beta}{2},\quad c=\frac{n}{2}, \tag{6.6}$$

where

$$\alpha=\frac{2k}{k+1},\quad \beta=\frac{n}{k}-2,\quad n\geq 2k,\quad k\geq 2. \tag{6.7}$$

Notice that $a>c>b>0$ and $a>b+1$ ($a=b+1$ when $k=1$) in our case. To verify (6.1), we need to demonstrate:

$$(-\Delta)^{\alpha/2}\left[F(a,b,c,-|x|^2)\right]^k\geq 0. \tag{6.8}$$

Let

$$\phi(x)=F(a,b,c,-|x|^2),\quad x\in\mathbb{R}^n. \tag{6.9}$$

By (4.4), ϕ clearly satisfies condition (1.6). Invoking Theorems 1.3 and 1.2, we see that (6.8) follows from condition (1.9):

$$([\phi(r)]^k)''+\frac{n-\alpha+1}{r}([\phi(r)]^k)'\leq 0,\quad r\geq 0.$$

By direct differentiation,

$$([\phi(r)]^k)''+\frac{n-\alpha+1}{r}([\phi(r)]^k)'=k[\phi(r)]^{k-1}\left[\phi''(r)+(k-1)\,\frac{[\phi'(r)]^2}{\phi(r)}+\frac{n-\alpha+1}{r}\phi'(r)\right].$$

Hence it suffices to verify the inequality

$$\phi''(r)+(k-1)\,\frac{[\phi'(r)]^2}{\phi(r)}+\frac{n-\alpha+1}{r}\phi'(r)\leq 0,\quad r>0. \tag{6.10}$$

Since $\phi(r)=F(-r^2)$, where $F(x)=F(a,b,c,x)$, we have $\phi'(r)=-2rF'(-r^2)$ and $\phi''(r)=4r^2F''(-r)-2F'(-r^2)$. It follows that the preceding inequality is equivalent to:

$$F''(-r^2)+(k-1)\,\frac{[F'(-r^2)]^2}{F(-r^2)}-\frac{n-\alpha+2}{2r^2}F'(-r^2)\leq 0,\quad r>0. \tag{6.11}$$

Letting $x=r^2$, $x>0$, and noticing that $\frac{n-\alpha+2}{2}=2c-a+1$ by (6.6), we rewrite the preceding inequality as:

$$F''(-x)+(k-1)\,\frac{[F'(-x)]^2}{F(-x)}-\frac{2c-a+1}{x}F'(-x)\leq 0,\quad x>0. \tag{6.12}$$

Using the hypergeometric equation ([AAR], p. 75):

$$x(1+x)F''(-x)-(c+(a+b+1)x)\,F'(-x)+abF(-x)=0, \tag{6.13}$$

we eliminate the second derivative in (6.12):

$$(k-1)[F'(-x)]^2+\left[\frac{c+(a+b+1)x}{x(1+x)}-\frac{2c-a+1}{x}\right]F'(-x)\,F(-x)-\frac{ab}{x(1+x)}[F(-x)]^2\leq 0.$$

Letting $\lambda = \frac{F'(-x)}{F(-x)}$, where $x \geq 0$, we reduce the preceding estimate to the quadratic inequality

$$x(1+x)(k-1)\lambda^2 + \left[(2a+b-2c)x + a - c - 1\right]\lambda - ab \leq 0, \tag{6.14}$$

for $x > 0$. From the differentiation formula ([AAR], p. 94),

$$F'(a,b,c,x) = \frac{ab}{c} F(a+1,b+1,c+1,x), \tag{6.15}$$

it follows:

$$\lambda = \frac{ab}{c} \frac{F(a+1,b+1,c+1,-x)}{F(a,b,c,-x)} > 0, \quad x > 0. \tag{6.16}$$

Solving inequality (6.14) for $\lambda > 0$, we obtain:

$$0 < \lambda \leq \frac{-(d_1 x - d_2) + \sqrt{(d_1 x - d_2)^2 + d_3 x(1+x)}}{2(k-1)x(1+x)}, \quad x \geq 0. \tag{6.17}$$

where

$$d_1 = 2a + b - 2c, \quad d_2 = c - a + 1, \quad d_3 = 4ab(k-1) = 4a(2c - a - b) \tag{6.18}$$

by (6.7). Equivalently,

$$0 < \lambda \leq \frac{2ab}{d_1 x - d_2 + \sqrt{(d_1 x - d_2)^2 + d_3 x(1+x)}}, \quad x > 0, \tag{6.19}$$

Using (6.16), we see that (6.14) is equivalent to:

$$\frac{F(a,b,c,-x)}{F(a+1,b+1,c+1,-x)} \geq \frac{d_1 x - d_2 + \sqrt{(d_1 x - d_2)^2 + d_3 x(1+x)}}{2c}, \quad x > 0. \tag{6.20}$$

We next use Pfaff's transformation ([AAR], Theorem 2.2.5, p. 68):

$$F(a,b,c,y) = (1-y)^{-b} F(c-a,b,c,\frac{y}{y-1}), \quad \text{for } y < 1. \tag{6.21}$$

Letting $y = -x$, $x \geq 0$, and applying (6.21) to both $F(a,b,c,-x)$ and $F(a+1,b+1,c+1,-x)$, we get

$$\frac{F(a,b,c,-x)}{F(a+1,b+1,c+1,-x)} = (1+x)\frac{F(c-a,b,c,\frac{x}{x+1})}{F(c-a,b+1,c+1,\frac{x}{x+1})}, \quad x > 0. \tag{6.22}$$

Letting $t = \frac{x}{x+1}$, we rewrite (6.20) in the equivalent form:

$$\frac{F(c-a,b,c,t)}{F(c-a,b+1,c+1,t)} \geq \frac{d_1 t - d_2(1-t) + \sqrt{(d_1 t - d_2(1-t))^2 + d_3 t}}{2c}, \quad 0 < t \leq 1, \tag{6.23}$$

where d_1, d_2, d_3 are given by (6.18). We note that since $(c-a)+b < c$ and $(c-a)+(b+1) < c+1$, it follows that the both hypergeometric functions in the ratio on the left-hand side are finite at $t = 1$. By (4.3),

$$F(c-a,b,c,1) = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}, \quad F(c-a,b+1,c+1,1) = \frac{\Gamma(c+1)\Gamma(a-b)}{\Gamma(a+1)\Gamma(c-b)}. \tag{6.24}$$

Consequently,

$$\frac{F(c-a,b,c,1)}{F(c-a,b+1,c+1,1)} = \frac{a}{c}. \tag{6.25}$$

In fact, since $a > b+1$ for $k \geq 2$, both hypergeometric functions in the ratio are twice continuously differentiable for $t \leq 1$, including the endpoint $t = 1$ when (see [AAR], Theorem 2.3.2, p. 78).

To prove (6.23), we express it in the form:

$$f(t) \geq g(t), \quad 0 \leq t \leq 1, \tag{6.26}$$

where f is the Gaussian ratio:

$$f(t) = \frac{F(c-a, b, c, t)}{F(c-a, b+1, c+1, t)}, \quad -\infty < t \leq 1, \tag{6.27}$$

and

$$g(t) = \frac{d_1 t - d_2(1-t) + \sqrt{(d_1 t - d_2(1-t))^2 + d_3 t}}{2c}, \quad t \geq 0. \tag{6.28}$$

The proof of (6.26) relies on the important observation that f is convex (in fact, even logarithmically convex, as is proved in the next section) while g is concave:

Lemma 6.1. *The function g defined by (6.28) is concave on* $[0, +\infty)$.

Proof. Let $h(t) = \sqrt{At^2 + Bt + C}$, then

$$h'(t) = \frac{2At + B}{2\sqrt{At^2 + Bt + C}}, \quad \text{and} \quad h''(t) = \frac{4AC - B^2}{4(At^2 + Bt + C)^{\frac{3}{2}}}.$$

Hence, h is concave if $B^2 - 4AC \geq 0$. Let $At^2 + Bt + C = (d_1 t - d_2(1-t))^2 + d_3 t$, where d_1, d_2, d_3 are defined by (6.18). Then clearly, $A = (d_1 - d_2)^2$, $B = d_3 - 2d_2(d_1 + d_2)$, $C = d_2^2$, and $At^2 + Bt + C \geq 0$ for $t \geq 0$ since $d_3 \geq 0$. Clearly,

$$B^2 - 4AC = [d_3 - 2d_2(d_1 + d_2)]^2 - 4(d_1 + d_2)^2 d_2^2 = d_3[d_3 - 4d_2(d_1 + d_2)],$$

where

$$d_3 - 4d_2(d_1 + d_2) = a(2c - a - b) - (c - a + 1)(a + b - c + 1) = (c + 1)(c - b - 1) \geq 0.$$

Notice that $c - b - 1 \geq 0$. Indeed, from (6.7) it follows that $n(k-1) \geq 2k$ since $n \geq 2k$ and $k \geq 2$. Hence,

$$c - b - 1 = \frac{n(k-1) - 2k + \alpha}{2k} > \frac{n(k-1) - 2k}{2k} \geq 0. \tag{6.29}$$

Thus, $B^2 - 4AC > 0$, and consequently h is concave. Since $g'' = \frac{1}{2c} h'' < 0$, it follows that g is concave as well. □

Since f is convex by Theorem 7.1 below, and g is concave by Lemma 6.1, it follows that the graph of f lies above the tangent line at $t = 1$, while the graph of g lies below the tangent line at $t = 1$. In other words, $f(t) \geq f(1) + (t-1)f'(1)$, while $g(t) \leq g(1) + (t-1)g'(1)$ for $0 \leq t \leq 1$. Recall that by (6.25), $f(1) = \frac{a}{c}$. Notice that $g(1) = \frac{a}{c}$ as well. Indeed,

$$g(1) = \frac{d_1 + \sqrt{d_1^2 + d_3}}{2c},$$

where

$$\begin{aligned} d_1^2 + d_3 &= a^2 - 2a(2c - a - b) + (2c - a - b)^2 + 4a(2c - a - b) \\ &= a^2 + 2a(2c - a - b) + (2c - a - b)^2 = (2c - b)^2. \end{aligned}$$

Since $2c > b$ by (6.7), we obtain:

$$\sqrt{d_1^2 + d_3} = 2c - b, \tag{6.30}$$

and consequently, by (6.18),

$$g(1) = \frac{d_1 + 2c - b}{2c} = \frac{2a + b - 2c + 2c - b}{2c} = \frac{a}{c}.$$

Thus, $f(1) = g(1)$, and to verify (6.26), it remains to show that $f'(1) \le g'(1)$. Let us deduce the formula:

$$f'(1) = \frac{a(a-c)}{c(a-b-1)}. \tag{6.31}$$

Invoking the differentiation formula (6.15), we obtain:

$$f'(t) = \frac{(c-a)b}{c}\,\frac{F(c-a+1,b+1,c+1,t)}{F(c-a,b+1,c+1,t)} - \frac{(c-a)(b+1)}{c+1}\,\frac{F(c-a+1,b+2,c+2,t)\,F(c-a,b,c,t)}{F(c-a,b+1,c+1,t)^2}.$$

Using (6.25), we deduce from the preceding formula:

$$f'(1) = \frac{(c-a)b}{c}\,\frac{F(c-a+1,b+1,c+1,1)}{F(c-a,b+1,c+1,1)} - \frac{a(c-a)(b+1)}{c(c+1)}\,\frac{F(c-a+1,b+2,c+2,1)}{F(c-a,b+1,c+1,1)}.$$

Applying Gauss's formula (6.24) for $F(a,b,c,1)$ ([AAR], Theorem 2.2.2, p. 66), we obtain

$$\begin{aligned} f'(1) &= \frac{(c-a)b}{c}\,\frac{\Gamma(c+1)\Gamma(a-b-1)\Gamma(a+1)\Gamma(c-b)}{\Gamma(a)\Gamma(c-b)\Gamma(c+1)\Gamma(a-b)} \\ &\quad - \frac{a(c-a)(b+1)}{c(c+1)}\,\frac{\Gamma(c+2)\Gamma(a-b-1)\Gamma(a+1)\Gamma(c-b)}{\Gamma(a+1)\Gamma(c-b)\Gamma(c+1)\Gamma(a-b)} \\ &= \frac{(c-a)ab}{c(a-b-1)} - \frac{a(c-a)(b+1)}{c(a-b-1)} = \frac{a(a-c)}{c(a-b-1)}. \end{aligned}$$

This proves (6.31).

On the other hand, by direct differentiation,

$$g'(t) = \frac{1}{2c}\left(d_1 + d_2 + \frac{2(d_1+d_2)(d_1 t - d_2(1-t)) + d_3}{2\sqrt{(d_1 t - d_2(1-t))^2 + d_3}}\right). \tag{6.32}$$

Hence,

$$g'(1) = \frac{1}{2c}\left(d_1 + d_2 + \frac{2(d_1+d_2)d_1 + d_3}{2\sqrt{d_1^2 + d_3}}\right). \tag{6.33}$$

We next show that, after simplification, we get:

$$g'(1) = \frac{a(c+1)}{c(2c-b)}. \tag{6.34}$$

To prove this, recall that $\sqrt{d_1^2 + d_3} = 2c - b$ by (6.30), and $d_1 = 2a + b - 2c = a - (2c - a - b)$, $d_1 + d_2 = a + b - c + 1$, and $d_3 = 4a(2c - a - b)$ by (6.18). It follows:

$$g'(1) = \frac{1}{2c}\left(a + b - c + 1 + \frac{(2a+b-2c)(a+b-c+1) + 2a(2c-a-b)}{2c-b}\right).$$

Simplifying further the right-hand side, we rewrite it as follows:

$$\begin{aligned} &\frac{(2c-b)(a+b-c+1) + (2a+b-2c)(a+b-c+1) + 2a(2c-a-b)}{2c(2c-b)} \\ &= \frac{2a(a+b-c+1) + 2a(2c-a-b)}{2c(2c-b)} = \frac{a(c+1)}{c(2c-b)}. \end{aligned}$$

This proves (6.34).

Now it is not difficult to see that $g'(1) \ge f'(1)$, i.e.,

$$\frac{a(a-c)}{c(a-b-1)} \le \frac{a(c+1)}{c(2c-b)}. \tag{6.35}$$

Indeed, by (6.7), $2c > b > 0$ and $a - b - 1 = \frac{n(k-1)}{2k} > 0$ for $k \geq 2$. Hence (6.35) is equivalent to:

$$(a-c)(2c-b) - (c+1)(a-b-1) \leq 0.$$

By factoring, we rewrite this as:

$$(a-c)(2c-b) - (c+1)(a-b-1) = ac - 2c^2 - ab + 2bc + c - a + b + 1)$$
$$= c(a-2c) - b(a-2c) - (a-2c) - (c-b-1) = (a-2c-1)(c-b-1) \leq 0.$$

By (6.29), $c - b - 1 \geq 0$. On the other hand, since $1 < \alpha < 2$, we have:

$$a - 2c - 1 = -\frac{n+2-\alpha}{2} < 0,$$

This proves (6.35), and consequently $g'(1) \geq f'(1)$, which yields (6.26).

These computations have been verified using *Mathematica*, which was also used to check that inequality (6.23) holds for many concrete values of k and n. In the next section, we will complete the proof of this estimate by showing that $f(t)$ is convex if $t \leq 1$.

Remark 6.2. When $k = 1$ and $\alpha = 1$, the function $u(x) = -(-\Delta)^{\frac{1}{2}}(1 + |x|^2)^{2-n}$, $n \geq 3$, is $\frac{1}{2}$-subharmonic, i.e. (6.1) holds, but inequality (1.9) with $s = \frac{1}{2}$ fails for r large. In other words, (1.9) is sufficient but not necessary for a C^2 radial function to be s-subharmonic.

Indeed, if $k = 1$, then $a = b + 1$, and consequently by (6.24) and (6.31), it follows that $f'(1) = +\infty$, where

$$f(t) = \frac{F(c-a, b, c, t)}{F(c-a, b+1, c+1, t)}, \quad -\infty < t \leq 1,$$

Note that inequality (6.12), and hence (6.14), is equivalent to (6.23), i.e., $f(t) \geq g(t)$ for $t \leq 1$, where $g(t)$ is defined by (6.28) with $d_3 = 0$. Since $f(1) = g(1) = \frac{a}{c}$, and $g'(1) < +\infty$ by (6.34), this contradicts $f'(1) = +\infty$.

7. Logarithmic convexity of the ratio of hypergeometric functions

Let $F(a, b, c, x) = {}_2F_1(a, b, c, x)$ denote the hypergeometric function. We refer to [AAR] and [MOS] for the relevant theory. For $c > b > 0$, it can be defined by ([AAR], p. 65):

$$F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{s^{b-1}(1-s)^{c-b-1}}{(1-sz)^a} ds, \tag{7.1}$$

in the complex plane with a cut from 1 to ∞ along the positive real axis. We will show in this section that the Gaussian ratio $\varphi(x) = \frac{F(a,b,c,x)}{F(a,b+1,c+1,x)}$ is logarithmically convex, and hence convex, for $x \in (-\infty, 1)$ if $c > b > 0$ and $-1 < a < 0$. Since $a + b < c$, it follows by (6.24) that φ is continuous at $x = 1$ as well, and hence φ is convex in $(-\infty, 1]$, including the end-point. Applying this to $f(x) = \frac{F(c-a,b,c,x)}{F(c-a,b+1,c+1,x)}$ where a, b, c are given by (6.6) so that $c > b > 0$, and $c - a = -\frac{\alpha}{2} \in (-1, 0)$, we will conclude the proof of Theorem 1.4.

The proof of the following theorem employs a method developed recently in [KS1] (see also [KS2]) to prove monotonicity and log-concavity in parameters a, b, c of generalized hypergeometric functions and their ratios.

Theorem 7.1. *Let $-1 < a < 0$, and $c > b > 0$. Then the function $\varphi(x) = \frac{F(a,b,c,x)}{F(a,b+1,c+1,x)}$ is logarithmically convex in $(-\infty, 1)$.*

Proof. Let $x \in (-\infty, 1)$. Let $f_1(x) = F(a, b, c, x)$ and $f_2(x) = F(a, b+1, c+1, x)$. Then $(\log \varphi)'(x) = \frac{f_1'(x)}{f_1(x)} - \frac{f_2'(x)}{f_2(x)}$, and

$$(\log \varphi)''(x) = \frac{f_1''(x)}{f_1(x)} - \frac{f_2''(x)}{f_2(x)} - \left(\left[\frac{f_1'(x)}{f_1(x)}\right]^2 - \left[\frac{f_2'(x)}{f_2(x)}\right]^2 \right)$$

$$= \frac{f_1''(x)f_2(x) - f_2''(x)f_1(x)}{f_1(x)\,f_2(x)} - \frac{[f_1'(x)f_2(x) - f_2'(x)f_1(x)]\cdot[f_1'(x)f_2(x) + f_2'(x)f_1(x)]}{[f_1(x)\,f_2(x)]^2}.$$

Clearly, $f_1(x) > 0$ and $f_2(x) > 0$ by (7.1). Since $a < 0$ and $b > 0, c > 0$, it follows by the differentiation formula (6.15):

$$f_1'(x) = \frac{ab}{c}\,F(a+1, b+1, c+1, x) < 0, \quad f_2'(x) = \frac{a(b+1)}{c+1}\,F(a+1, b+2, c+2, x) < 0,$$

and consequently,

$$f_1'(x)f_2(x) + f_2'(x)f_1(x) < 0, \quad -\infty < x < 1.$$

Hence to show that $(\log\varphi)''(x) > 0$, it suffices to prove the following inequalities:

$$f_1'(x)f_2(x) - f_2'(x)f_1(x) > 0, \quad f_1''(x)f_2(x) - f_2''(x)f_1(x) > 0, \quad -\infty < x < 1. \tag{7.2}$$

It will be more convenient to make a substitution $t = -x$, and work with $g_1(t) = f_1(-t)$, $g_2(t) = f_2(-t)$ for $t \in (-1, +\infty)$. Then the preceding inequalities are equivalent to:

$$g_1'(t)g_2(t) - g_2'(t)g_1(t) < 0, \quad g_1''(t)g_2(t) - g_2''(t)g_1(t) > 0, \quad -1 < t < +\infty. \tag{7.3}$$

We next employ the integral representation (7.1) to express both g_1 and g_2 in the form of Stieltjes-type transformations:

$$g_1(t) = \int_0^1 s^{b-1}\,(1+st)^{-a}\,w(s)\,ds, \quad g_2(t) = \frac{c}{b}\int_0^1 s^{b}\,(1+st)^{-a}\,w(s)\,ds, \quad -1 < t < +\infty,$$

where

$$w(s) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}(1-s)^{c-b-1}, \quad 0 < s < 1,$$

is a positive weight function. Then we have:

$$g_1'(t) = -a\int_0^1 s^{b}\,(1+st)^{-a-1}\,w(s)\,ds, \quad g_2'(t) = -a\,\frac{c}{b}\int_0^1 s^{b+1}\,(1+st)^{-a-1}\,w(s)\,ds,$$

$$g_1''(t) = a(a+1)\int_0^1 s^{b+1}\,(1+st)^{-a-2}\,w(s)\,ds, \quad g_2''(t) = a(a+1)\,\frac{c}{b}\int_0^1 s^{b+2}\,(1+st)^{-a-2}\,w(s)\,ds,$$

where $-1 < t < +\infty$.

To prove the first inequality in (7.3), i.e., $g_1'(t)g_2(t) < g_2'(t)g_1(t)$, we rewrite it in the equivalent form:

$$-a\,\frac{c}{b}\left(\int_0^1 s^{b}\,(1+st)^{-a-1}\,w(s)\,ds\right)\left(\int_0^1 s^{b}\,(1+st)^{-a}\,w(s)\,ds\right)$$
$$< -a\,\frac{c}{b}\left(\int_0^1 s^{b+1}\,(1+st)^{-a-1}\,w(s)\,ds\right)\left(\int_0^1 s^{b-1}\,(1+st)^{-a}\,w(s)\,ds\right).$$

Similarly, the second inequality in (7.3), i.e., $g_1''(t)g_2(t) > g_2''(t)g_1(t)$, is equivalent to:

$$a(a+1)\,\frac{c}{b}\left(\int_0^1 s^{b+1}\,(1+st)^{-a-2}\,w(s)\,ds\right)\left(\int_0^1 s^{b}\,(1+st)^{-a}\,w(s)\,ds\right)$$
$$> a(a+1)\,\frac{c}{b}\left(\int_0^1 s^{b+2}\,(1+st)^{-a-2}\,w(s)\,ds\right)\left(\int_0^1 s^{b-1}\,(1+st)^{-a}\,w(s)\,ds\right).$$

Fix $t \in (-1, +\infty)$. Since $c > b > 0$, and $-1 < a < 0$, the first inequality above can be expressed in the form (see [KS1], p. 341):

$$\left(\int_0^1 h(s)\,p(s)\,ds\right)\left(\int_0^1 q(s)\,p(s)\,ds\right) < \left(\int_0^1 q(s)\,h(s)\,p(s)\,ds\right)\left(\int_0^1 p(s)\,ds\right), \tag{7.4}$$

where $h(s) = \frac{s}{1+st}$, $p(s) = s^{b-1}(1+st)^{-a}w(s)$, $q(s) = s$ and $w(s) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}(1-s)^{c-b-1}$.

Analogously, the second inequality is equivalent to:

$$\left(\int_0^1 h_1(s)\,p(s)\,ds\right)\left(\int_0^1 q(s)\,p(s)\,ds\right) < \left(\int_0^1 q(s)\,h_1(s)\,p(s)\,ds\right)\left(\int_0^1 p(s)\,ds\right), \tag{7.5}$$

where $h_1(s) = \frac{s^2}{(1+st)^2}$, $p(s)$, $q(s)$ and $w(s)$ are as above. Since $p(s) > 0$, and the functions $q(s)$, $h(s)$, and $h_1(s)$ are increasing on $(0,1)$ for any fixed $t > -1$, both (7.4) and (7.5) follow from Chebyshev's inequality for monotone functions (see [HLP]). □

Remark 7.2. The condition $c > b > 0$ in Theorem 7.1 can be extended to $c > \min(a,b) > 0$ using the symmetry of $F(a,b,c,x)$ in a and b.

Remark 7.3. Theorem 7.1 holds for the ratio of hypergeometric functions ${}_{q+1}F_q\left((a,\mathbf{b}),\mathbf{c},x\right)$ in place of ${}_2F_1(a,b,c,x)$, where $-1 < a < 0$ and $c_k > b_k > 0$, $k = 1,2,\ldots,q$, $q \geq 2$.

The proof of the logarithmic convexity of the ratio of ${}_{q+1}F_q$ is similar to the proof of Theorem 7.1 using the corresponding Stieltjes type integral representation ([KS1], Lemma 1).

Corollary 7.4. *Let $0 < a - c < 1$, and $c > b > 0$. Then the ratio $\frac{F(a,b,c,x)}{F(a+1,b+1,c+1,x)}$ is convex in $(-\infty, 1)$.*

Proof. Let $\varphi_1(-x) = \frac{F(a,b,c,x)}{F(a+1,b+1,c+1,x)}$, $x < 1$. By (6.22),

$$\varphi_1(x) = (1+x)\,\varphi_0\left(\frac{x}{x+1}\right), \quad x > -1,$$

where

$$\varphi_0(y) = \frac{F(c-a,b,c,y)}{F(c-a,b+1,c+1,y)}, \quad y < 1.$$

We deduce:

$$\varphi_1'(x) = \varphi_0\left(\frac{x}{x+1}\right) + \varphi_0'\left(\frac{x}{x+1}\right)\frac{1}{1+x}, \quad \varphi_1''(x) = \varphi_0''\left(\frac{x}{x+1}\right)\frac{1}{(x+1)^3} \geq 0,$$

where φ_0 is logarithmically convex by Theorem 7.1, and consequently convex, so that $\varphi_0''(y) \geq 0$, $y < 1$. Then, obviously, $\varphi_1(-x)$ is convex as well. □

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Dipartimento di Matematica, dell'Università di Bologna, Piazza di Porta S. Donato, 5, 40126, Bologna, Italy

E-mail address: fausto.ferrari@unibo.it

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: verbitskyi@missouri.edu